Term Project
Neutron Star Models

Eric Hsiao

March 26, 2004

1 Introduction

In 1939, Oppenheimer and Volkoff presented a paper on the equilibrium masses of neutron stars, in which they noted that the maximum allowable mass is $\sim 0.7M_{\odot}$. Since then, the masses of observed neutron stars were measured to be well above this figure, indicating that nuclear repulsion has to be considered for stars of small radii. In this project, neutron star models were constructed using five different equations of state, each based on a different hypothesis of nuclear interaction. The resulting configurations of the maximum allowable mass were then compared.

2 Equations of hydrostatic equilibrium

The relativistic equations of hydrostatic equilibrium are

$$\frac{dM_r}{dr} = 4\pi r^2 \frac{\epsilon}{c^2} \quad \text{and} \quad \frac{dP}{dr} = -\frac{G(\epsilon/c^2 + P/c^2)(M_r + 4\pi r^3 P/c^2)}{r(r - 2GM_r/c^2)}.$$

New units for mass u and length r are defined as follows:

$$[r] = \frac{1}{\pi}\sqrt{\frac{hG}{c^3}}\left(\frac{h}{mc}\right)\left(\frac{c^2}{Gm}\right) = 13.6831km,$$

$$[u] = \frac{1}{\pi}\left(\frac{hc}{G}\right)^{\frac{3}{2}}\frac{1}{m^2} = 9.2661M_{\odot},$$

where m is the neutron mass. The equations of hydrostatic equilibrium using these new units become the expressions of Oppenheimer and Volkoff:

$$\frac{du}{dr} = 4\pi\epsilon r^2 \quad \text{and} \quad \frac{dt}{dr} = -4\pi r\frac{P+\epsilon}{1-\frac{2u}{r}}\left(P + \frac{u}{4\pi r^3}\right)\left(\frac{dp}{dt}\right)^{-1},$$

where t is a parameter to be defined later.

The boundary condition at the centre of the star is defined at $t = t_0$, where $r = 0$ and $u = 0$. For the computer program, $r = 0$ presents a problem in the second equation of hydrostatic equilibrium. To resolve the problem, a small value of radius $r = r_c$ was used. The following expression was then used to obtain the corresponding boundary value for the mass u_c at the centre,

$$u_c = \frac{4\pi}{3} r_c^3 \epsilon(t_0), \quad \text{where } \frac{du}{dr} = 4\pi\epsilon(t) r^2.$$

3 Equations of state

Five different equations of state (EOS) were used to compute the neutron star models. The first four were presented by Inman in 1964. The last and more modern equation of state was presented by Douchin and Haensel in 2001.

The energy density ϵ and pressure P can each be written as the sum of its kinetic and potential component, subscript T and ν respectively:

$$\epsilon = \epsilon_T + \epsilon_\nu \quad \text{and} \quad P = P_T + P_\nu.$$

For completely degenerate neutron gas, the kinetic pressure is

$$P_T = \frac{\pi m^4 c^5}{3h^3} f(x), \quad \text{where } f(x) = x(2x^2 - 3)\sqrt{x^2 + 1} + 3\sinh^{-1} x.$$

The parameter x can be written in terms of the parameter t by the relation $x = \sinh t/4$, then $f(x)$ becomes

$$f(t) = \frac{1}{4}\sinh t - 2\sinh\frac{t}{2} + \frac{3}{4}t.$$

The kinetic energy density can be expressed as $\epsilon_T = \epsilon_{kin} + \epsilon_{rest}$, where

$$\epsilon_{kin} = \frac{\pi m^4 c^5}{3h^3}[8x^3(\sqrt{x^2 + 1} - 1) - f(x)],$$

$$\epsilon_{rest} = nmc^2 = \frac{\pi m^4 c^5}{3h^3}[8x^3], \quad \text{for } n = \frac{8\pi m^3 c^3}{3h^3} x^3.$$

Then the kinetic energy density can be written in terms of the parameter x:

$$\epsilon_T = \epsilon_{kin} + \epsilon_{rest} = \frac{\pi m^4 c^5}{3h^3} h(x), \quad \text{where}$$

$$h(x) = [8x^3(\sqrt{x^2 + 1} - 1) - f(x) + 8x^3] = \frac{\pi m^4 c^5}{3h^3}[8x^3\sqrt{x^2 + 1} - f(x)],$$

and in terms of the parameter t:

$$h(t) = \frac{3}{4}(\sinh t - t).$$

The energy density ϵ and pressure P have the same unit of erg/cm^{-3}. If we now define a new unit of energy density

$$[\epsilon] = [P] \equiv \frac{1}{6}\frac{mc^2}{\frac{4}{3}\pi(\frac{h}{2\pi mc})^3} = \frac{\pi^2 m^4 c^5}{h^3} = 6.4658(10^{36})\frac{erg}{cm^3},$$

then the kinetic energy density and pressure can be written in this new unit as:

$$\epsilon_T = \frac{1}{4\pi}(\sinh t - t) \quad \text{and} \quad P_T = \frac{1}{12\pi}\left(\sinh t - 8\sinh\frac{t}{2} + 3t\right).$$

The first four EOS presented by Inman uses the same kinetic energy density and pressure shown above.

3.1 EOS 1

EOS 1 is the case of neutrons without interactions, presented by Oppenheimer and Volkoff in 1939:

$$\epsilon_\nu = P_\nu = 0.$$

3.2 EOS 2

EOS 2 is derived from a nuclear potential given by Skyrme in 1959, based on the many-body theory of nuclear matter.

$$\epsilon_\nu = \frac{1}{4\pi}\left(23.9\sinh^8\frac{t}{4} - 10.1\sinh^6\frac{t}{4}\right),$$

$$P_\nu = \frac{1}{4\pi}\left(39.9\sinh^8\frac{t}{4} - 10.1\sinh^6\frac{t}{4}\right).$$

3.3 EOS 3

EOS 3 is based on Zel'dovich's "hard core" hypothesis of nuclear interactions in 1959. Strong nuclear repulsion is present at small distances. The parameter $\frac{\alpha}{\beta}$ was taken as 1 in the model construction.

$$\epsilon_\nu = \frac{16}{9\pi^2}\frac{\alpha}{\beta}\sinh^5\frac{t}{4},$$

$$P_\nu = \frac{32}{27\pi^2}\frac{\alpha}{\beta}\sinh^5\frac{t}{4}.$$

3.4 EOS 4

EOS 4 is based on Zel'dovich's equations of state in 1961 that is compatible with the theory of relativity. The parameter γ was taken as 3 in the model construction.

$$\epsilon_\nu = P_\nu = \frac{16}{9\pi^2}\gamma \sinh^6 \frac{t}{4}.$$

3.5 EOS 5

The neutron star crust and liquid core equations of state were calculated and presented by Douchin and Haensel as data sets in nucleon number density n in fm^{-3}, density ρ in g/cm^{-3}, pressure P in erg/cm^{-3} and a dimensionless adiabatic index Γ. It is convenient to convert these quantities to t, ϵ, P and $\frac{dP}{dt}$ in the units used in the first four equations of state. The adiabatic index is defined as

$$\Gamma = \frac{n}{P}\frac{dP}{dn}, \quad \text{where } n = \frac{8\pi m^3 c^3}{3h^3}x^3 = \frac{8\pi m^3 c^3}{3h^3}\sinh^3 \frac{t}{4}.$$

The parameter t can then be written in terms of the nucleon number density:

$$t = 4\sinh^{-1}\left[\left(\frac{3h^3}{8\pi m^3 c^3}\right)^{\frac{1}{3}} n^{\frac{1}{3}}\right].$$

Knowing t for each data point, $\frac{dP}{dt}$ can be determined from given Γ and P:

$$\frac{dP}{dt} = \frac{dP}{dn}\frac{dn}{dt} = \frac{P}{n}\Gamma\frac{dn}{dt} = \frac{3}{4}\frac{\Gamma P}{\tanh t/4}.$$

4 Numerical Methods

The numerical methods used in the neutron star models programs are included in this section.

4.1 Runge-Kutta method

To solve the two equations of hydrostatic equilibrium simultaneously, the fourth-order Runge-Kutta method was used. Let

$$\frac{du}{dr} = f(r,t) \quad \text{and} \quad \frac{dt}{dr} = g(r,u,t).$$

Then the Runge-Kutta method adds four incremental components symmetrically to solve for u and t:

$$\begin{array}{ll}
u_{n+1} = u_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) & t_{n+1} = t_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\
k_1 = hf(r_n, t_n) & l_1 = hg(r_n, u_n, t_n) \\
k_2 = hf(r_n + \frac{h}{2}, t_n + \frac{l_1}{2}) & l_2 = hg(r_n + \frac{h}{2}, u_n + \frac{k_1}{2}, t_n + \frac{l_1}{2}) \\
k_3 = hf(r_n + \frac{h}{2}, t_n + \frac{l_2}{2}) & l_3 = hg(r_n + \frac{h}{2}, u_n + \frac{k_2}{2}, t_n + \frac{l_2}{2}) \\
k_4 = hf(r_n + h, t_n + l_3) & l_4 = hg(r_n + h, u_n + k_3, t_n + l_3).
\end{array}$$

The parameter h is the r increment and is associated with the error of the calculation as $\sim h^3$.

4.2 Interpolation

For EOS 5, a set of data points was given instead of analytical functions. The data was in the form x_i and y_i, where i is the index number. To obtain reasonable value of y for a given value of x, the index i that gives the closest values between $x(i)$ and x was first located. The interpolation formula then estimates the values of y with the data points indexed $i-1$, i and $i+1$:

$$y(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}y_{i-1} + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}y_i + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}y_{i+1}.$$

5 Results

The programs used for the model constructions were attached. The Runge-Kutta calculations run from an initial value of t_0 and stops when $t = 0.001t_0$, and the values of r and u were recorded at this point.

The results were plotted in Figure 1. EOS 1 model, without accounting for nuclear interactions, had the lowest maximum allowable mass $\sim 0.7M_\odot$. When nuclear interactions were considered, the maximum allowable mass increases to over $2M_\odot$ in the case of EOS 5.

Figure 2 plots the mass of the neutron stars against their central densities. It was shown that the model is dynamically unstable for the part of figure where $\frac{dM}{d\rho_c} < 0$. Compact objects with mass greater than the maximum allowable mass cannot be neutron stars. The maximum mass for a neutron star were lifted from $0.7M_\odot$ when nuclear repulsion is considered. The configuration of maximum mass of static neutron stars for each equation of state was summarized in the following table:

eos	t_0	$M[M_\odot]$	$R[km]$	$\rho_c[10^{14}\ g/cm^3]$	$P_c[10^{36}\ erg/cm^3]$
1	3.03	0.710	9.18	41.7	0.359
2	2.93	1.70	7.99	44.4	2.38
3	2.47	1.14	13.2	21.1	0.214
4	2.45	1.59	12.8	21.9	0.380
5	2.58	2.05	9.88	28.3	1.36

Figure 3 illustrates the properties of the equations of state in a $\log_{10} P$ vs. $\log_{10} \rho$ plot. The EOS 2 graph has a similar shape to the more modern EOS 5. The other three equations of state have almost straight lines with different slopes.

6 Conclusion

The equations of state that include the effects of nuclear interactions lift the maximum allowable mass of a neutron star from the Oppenheimer-Volkoff mass of $0.7M_{\odot}$. The more modern EOS 5 gives a maximum mass of $2.05M_{\odot}$.

References

Chandrasekhar, S. 1957, An Introduction to the Study of Stellar Structure (Dover)

Douchin, F. & Haensel, P. 2001, A&A, 380, 151

Inman, C. L. 1965, ApJ, 141, 187

Matthews, J. & Walker, R. L. 1970, Mathematical Methods of Physics (Benjamin-Cummings)

Oppenheimer, J. R. & Volkoff, G. M. 1939, Phys. Rev., 55, 374